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The large N limit of QCD and the collective field of the Hitchin fibration

Marco Bochicchio

INFN Sezione di Roma, Dipartimento di Fisica
Università di Roma "La Sapienza"
Piazzale Aldo Moro 2 , 00185 Roma
E-mail: Marco.Bochicchio@roma1.infn.it

Abstract: By means of a certain exact non-abelian duality transformation, we show that there is a natural embedding, dense in the sense of the distributions in the large N limit, of parabolic Higgs bundles of rank N on a fiber two-dimensional torus into the QCD functional integral, fiberwise over the base two-dimensional torus of the trivial elliptic fibration on which the four-dimensional theory is defined. The moduli space of parabolic Higgs bundles of rank N is an integrable Hamiltonian system, that admits a foliation by the moduli of holomorphic line bundles over N-sheeted spectral covers (or, what is the same, over a space of N gauge-invariant polynomials), the Hitchin fibration. According to Hitchin, the Higgs bundles can be recovered from the spectral covers and the line bundles. If the N invariant polynomials together with the abelian connection on the line bundles are chosen as the $N+1$ collective fields of the Hitchin fibration, all the entropy of the functional integration over the moduli of the Higgs bundles is absorbed, in the large N limit, into the Jacobian determinant of the change of variables to the collective fields of the Hitchin fibration. Hence, the large N limit is dominated by the saddle-point of the effective action as in vector-like models.

Keywords: Duality in Gauge Field Theories, 1/N Expansion, Differential and Algebraic Geometry.

Contents

1. Introduction

The aim of this paper, that is a brief summary of a more extended one, is to gain control over the large N limit of QCD in four dimensions [1].

We consider QCD on a four-dimensional torus, that we think as a two- dimensional fiber torus T_f^2, with complex coordinates $(z, \bar{z})$, trivially fibered over a two-dimensional base torus T_b^2, with complex coordinates $(u, \bar{u})$.

The basic idea consists in changing variables, as in a fiberwise duality transformation [2], from the four-dimensional gauge connection $(A_z, A_{\bar{z}}, A_u, A_{\bar{u}})$ to the cotangent space of unitary connections on the fiber torus, $T^*\mathcal{A}$, fiberwise over the base torus, with coordinates $(A_z, A_{\bar{z}}, \Psi_z, \Psi_{\bar{z}})$. $(\Psi_z, \Psi_{\bar{z}})$ are the components of a one-form, Ψ, the Higgs current, on the fiber torus fiberwise, that label the cotangent directions to the unitary connections $(A_z, A_{\bar{z}})$.

$T^*\mathcal{A}$ is an infinite-dimensional hyper-Kahler manifold on which the gauge group, $SU(N)$, acts by Hamiltonian vector fields with respect to all the three symplectic forms [3, 4].

We then quotient, by the action of the gauge group, over a densely chosen subset of $T^*\mathcal{A}$ [5], in order to make the quotient a separable manifold.

The quotient turns out to be the inductive limit, with respect to the partial ordering defined by inclusion of any divisor, of the moduli space of parabolic (with respect to a given divisor) $K(D)$ pairs [6] on the fiber torus fiberwise. This moduli space is a Kahler but not a hyper-Kahler manifold. However, requiring the residue of the Higgs field to be nilpotent, a hyper-Kahler reduction is obtained, corresponding to the moduli of parabolic Higgs bundles [6, 7]. The last one is a closed subspace of the moduli of parabolic $K(D)$ pairs. The dimensions of the two mentioned moduli spaces are of order N^2. Therefore the two cases present equivalent difficulties from the point of view of solving the leading large N limit. In fact, they differ only at sub-leading order $\frac{1}{N}$. We will not attempt to give a physical interpretation of this fact in this paper.

So far we have transformed, on a densely defined subset, the unitary monodromies along four-dimensional curves, a local system with order of N^2 degrees of freedom per point, into another local system, with the same order of N^2 degrees of freedom per point, the moduli of parabolic $K(D)$ pairs or the moduli of Higgs bundles fiberwise. Indeed, the last one is labelled by the monodromies, with values in the complexified gauge group, around arbitrary points on the fiber torus fiberwise [3, 8].

We now make the last crucial step, that allows us to put the QCD partition function in the form of a vector-like model.

According to Hitchin [9, 10, 11, 12], the moduli of $K(D)$ pairs or of Higgs bundles is an integrable Hamiltonian system, foliated by the moduli of line bundles over spectral branched coverings. These spectral covers are obtained from a basis of N gauge-invariant polynomials in Ψ_z via the characteristic equation. Remarkably, all the information of the parabolic Higgs or $K(D)$ bundles on the fiber torus can be reconstructed from a spectral covering of the torus and a line bundle on the cover [9, 10, 11, 12].

We are thus recovering a rank N local system on the fiber torus (with a parabolic divisor) from a rank 1 local system (with a parabolic divisor) on a N-sheeted branched covering of the fiber torus.

Yet, the last system is completely determined by a collection of $N+1$ local fields, an abelian $U(1)$ gauge connection, that determines the line bundle on the cover, and N meromorphic differentials, holomorphic away from the divisor, that determine the spectral cover.

We call these fields the collective [13] fields of the Hitchin fibration.

Thus, by changing variables to the collective fields of the Hitchin fibration, the QCD partition function looks like the one of a vector-like model.

As a consequence, the large N limit is dominated by the saddle point of the effective action, that now includes the Jacobian determinant of the change of variables from the moduli of parabolic Higgs bundles to the collective field of the Hitchin fibration.

2. The main formulae

Our starting point is the $SU(N)$-YM functional integral:

$$Z = \int \exp\left[-\frac{1}{4g^2}\int \sum_{\mu\nu} Tr\left(F_{\mu\nu}^2\right) d^4x\right] DA\,,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\,. \tag{2.1}$$

The four-dimensional space-time is chosen to be a four-dimensional Euclidean torus T^4 that it is thought as a trivial elliptic fibration of a two-dimensional fiber torus T_f^2 over another two-dimensional base torus T_b^2. The fiber torus has real coordinates

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(x_0, x_1) while the base torus has coordinates (x_2, x_3). We choose a complex structure on the fiber torus with complex coordinates $(z, \bar{z})$ and a complex structure on the base with coordinates $(u, \bar{u})$. Tangent directions to the fiber are indicated by $\|$, while directions tangent to the base with $\perp$. With this notation, introducing the auxiliary variable $E_\perp$, the partition function can be written as a Gaussian integral over $E_\perp$:

$$Z = \int \exp\left[-\frac{N}{2\lambda}\sum_{\|\perp}\int Tr\left(F_\|^2 + E_\perp^2 - 2iE_\perp F_\perp + F_{\|\perp}^2\right) d^4x\right] DA_\| DA_\perp DE_\perp. \quad (2.2)$$

The integration over the components of the connection transverse to the fiber, $A_\perp$, is Gaussian and can be performed explicitly, giving as a result the effective action for $E_\perp$ and $A_\|$ [2]. From now on, $A_\|$ will be indicated simply as A and $E_\perp$ as E unless stated otherwise. This gives for the partition function:

$$\begin{aligned} Z = \int \exp\left[-\frac{N}{2\lambda}\int Tr\left(F_A^2 + (\partial_\perp A)^2 + E^2 + \cdots\right) d^4x\right] \times \\ \times\, Det\left[-\Delta_A - i\, ad_E\right]^{-1} DADE\,, \end{aligned} \quad (2.3)$$

where the dots indicate non-local terms, some of which will be made to vanish later on, by an appropriate choice of the gauge-fixing condition, and the sum over the appropriate space-time indices is understood. Δ_A is the two-dimensional scalar Laplacian in the background of the connection A and ad_E the adjoint action of the Lie algebra valued field E. At this point, we make a change of variables, that allows us to embed $T^*\mathcal{A}$ into the QCD functional integral. We simply set

$$E_\perp = \epsilon_{\perp\|} F_\|(A_D)\,, \quad (2.4)$$

where $\epsilon_{\perp\|}$ is the rank four normalized antisymmetric tensor. We call this change of variables a fiberwise duality transformation, because it has the structure of a duality transformation restricted to the fiber [2]. Indeed the complete duality transformation in four dimensions [14, 15] would involve solving the Bianchi identities for the dual field strength:

$$\left[\partial_\mu + iA_\mu^D, \tilde{K}_{\mu\nu}\right] = 0\,, \quad (2.5)$$

that imply

$$K_{\mu\nu} = F_{\mu\nu}(A_D)\,, \quad (2.6)$$

up to some field-strength copy problem [16, 17]. To implement this change of variables in the functional integral we employ the resolution of the identity, by means of the Faddeev-Popov trick:

$$1 = \Delta_D(E)\int \delta[E - F(A_D)] DA_D\,, \quad (2.7)$$

where $F(A_D)$ is a dual curvature two-form associated to the dual connection one-form A_D. It is convenient for our purposes to decompose A_D into A and an arbitrary one-form Ψ:

$$A_D = A + \Psi \,. \tag{2.8}$$

Correspondingly, the Faddeev-Popov trick becomes, after shifting the A_D integration by A:

$$1 = \Delta_D(E, A) \int \delta[E - F(A + \Psi)] D\Psi \,. \tag{2.9}$$

We may consider the fields (A, Ψ) as the coordinates of $T^*\mathcal{A}$, the cotangent space of unitary connections, A, on the fiber torus. (A, Ψ) are four-dimensional fields as functions on space-time, but they belong to $T^*\mathcal{A}$ fiberwise. The QCD functional integral then becomes:

$$\begin{aligned} Z = \int \exp\left[-\frac{N}{2\lambda}\int Tr\left(F_A^2 + (\partial_\perp A)^2 + E^2 + \cdots\right) d^4x\right] \times \\ \times\, Det\left[-\Delta_A - i\, ad_E\right]^{-1} \Delta_D(A, \Psi)\delta[E - F_{A+\Psi}] DAD\Psi DE \,. \end{aligned} \tag{2.10}$$

The DE integration can now be performed by a convenient choice of the gauge, that has also the advantage of eliminating some of the non-local terms in the functional integral that were indicated by the dots. The gauge choice is [2]:

$$d^*A + E = 0 \,, \tag{2.11}$$

where d^* indicates the two-dimensional divergence of A. Inserting this gauge condition and the corresponding Faddeev-Popov determinant Δ_{FP} and performing the DE integration we get:

$$\begin{aligned} Z = \int \exp\left[-\frac{N}{2\lambda}\int Tr\left(F_A^2 + (d^*A)^2 + (\partial_\perp A)^2 + \cdots\right) d^4x\right] \times \\ \times\, Det[-\Delta_A - i\, ad_{F_{A+\Psi}}]^{-1} \Delta_D(A, \Psi)\Delta_{FP} \times \\ \times\, \delta[d^*A + F_{A+\Psi}] DAD\Psi \,. \end{aligned} \tag{2.12}$$

This gives the desired embedding of $T^*\mathcal{A}/G$ fiberwise in the functional integral.

Analytically our problem is, apparently, as difficult as the original one. Instead of integrating over the connection one-form in four dimensions (minus a gauge-fixing condition) we integrate over $T^*\mathcal{A}$ fiberwise (minus a gauge-fixing condition). After gauge-fixing, in both cases, the effective action is, in general, non-local.

Our main step will be, nevertheless, to use algebraic geometry to give the quotient, $T^*\mathcal{A}/G$, of $T^*\mathcal{A}$ by the gauge group $G = SU(N)$, an explicit meaning as a moduli space. The fact that this moduli space will turn out to be an integrable system will be essential for our approach to the large N limit. Had we tried to

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construct directly, in some sense, the moduli space of four- dimensional bundles, without integrating out the transverse polarizations of the connection, we would not have obtained such integrability properties.

To understand the moduli space, we must understand the structure of the gauge orbits in $T^*\mathcal{A}$. The gauge group acts on the fields A, Ψ by hamiltonian vector fields with respect to the three symplectic forms [3]. The corresponding moment maps are:

$$\begin{aligned} \mu^0 &= F_A - i[\Psi, \Psi]\,, \\ \mu &= \bar{\partial}_A \psi\,, \\ \bar{\mu} &= \partial_A \bar{\psi}\,, \end{aligned} \tag{2.13}$$

where in this formula ψ and $\bar{\psi}$ refer to the z and $\bar{z}$ components of the one-form $\Psi = \psi + \bar{\psi}$. General principles of hyper-Kahler geometry require, for the quotient under the action of the gauge group to be separable and hyper-Kahler, the moment map to be central. For $SU(N)$ on a compact Riemann surface this forces the moment map to vanish, unless the structure group of the cotangent bundle is reducible. This is a too restrictive constraint in the functional integral. However, we can consider bundles with a parabolic flag structure and weights on an arbitrary divisor D [18, 11]. Set in another way, we put coadjoint orbits on the given divisor associated to the parabolic structure [19]. This amounts to add, to the three moment maps for the gauge connection and the Higgs current, delta-like singularities on the divisor. A collective field for the parabolic Higgs bundles can then be introduced in the functional integral by means of the resolution of the identity [5]:

$$\begin{aligned} 1 = \lim_{|D|\to\infty} \int \delta\left(\mu^0 - \frac{1}{|D|}\sum_p \mu_p^0 \delta_p\right) \times \\ \times\, \delta\left(\mu - \frac{1}{|D|}\sum_p \mu_p \delta_p\right) \delta\left(\bar{\mu} - \frac{1}{|D|}\sum_p \bar{\mu}_p \delta_p\right) \times \\ \times \prod_p d\mu_p^0 d\mu_p d\bar{\mu}_p d^2 z_p\, J[\mu_0, \mu, \bar{\mu}]\,, \end{aligned} \tag{2.14}$$

where δ_p is the two-dimensional delta-function localized at z_p and J a compensating Jacobian determinant. This collective field for the moment maps is the analogue, in our gauge-theoretic setting, of the collective field for the density of the eigenvalues, introduced long ago for solving the large N limit of matrix models [13]. All the variables in Eq. (2.14) are functions of the coordinates $(u, \bar{u})$ on the base torus. We will leave implicit this functional dependence in all the following formulae. The levels of the moment maps are dense in the sense of the distributions, as the divisor gets larger and larger. μ_p^0 is a traceless hermitian matrix while μ_p and $\bar{\mu}_p$ live in the Lie algebra $sl(N)$. There is still the freedom of making $SU(N)$ gauge transformations. This was implemented, in the previous discussion, by the choice of the gauge-fixing.

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A deeper justification for considering moment maps valued in the distributions relies on the following facts. In two dimensions (or fiberwise, as in our case) there is a correspondence between unitary bundles and holomorphic ones [20]. In general, to study the moduli problem, it is more convenient the holomorphic language, since geometric invariant theory tells us when a moduli space exists as a separable manifold. A necessary and often sufficient condition is stability in the sense of Mumford [21]. Now, in a remarkable theorem, Narasimhan and Seshadri [21] proved that semi-stable holomorphic bundles on a compact Riemann surface arise as unitary projective representation of the fundamental group of the surface (or unitary representations of the once-punctured surface). Stable bundles correspond to irreducible representations. In addition, the moduli of non-stable bundles cannot be separable in general [21]. The Narasimhan-Seshadri result has been extended to the unitary non-compact case by Metha and Seshadri [18], who proved that stable holomorphic bundles with a parabolic flag structure (of parabolic degree zero) arise as unitary representation of the fundamental group of a compact Riemann surface with marked points and fixed monodromy conjugacy classes around the marked points. There is, moreover, a correspondence between the weights of the parabolic structure and the unitary eigenvalues of the monodromy, and between the flag structure and the eigenspaces of the eigenvalues of the monodromy as well. More simply, the moduli of these bundles can be described, in the language of moment maps and symplectic reduction, as a quotient of the space of unitary connections with coadjoint orbits under the action of the gauge group [19]. The computation of the analytic torsion in YM_2 fits into this symplectic framework [22]. Later on, an analogous computation, in the hyper-Kahler case of Hitchin bundles, will play some role in this paper. Hitchin extended the Narasimhan-Seshadri result in another direction. He answered the question to what correspond non-unitary representations of the fundamental group of a compact surface [3]. The answer is stable (semi-stable) Higgs bundles. Given a decomposition of a non-unitary flat connection into a unitary one plus a one-form valued in the complement of the compact generators, it is possible to choose the one-form in such a way that it is harmonic with respect to the unitary connection [8]. The corresponding decomposition of the non-hermitian curvature furnishes the three moment maps for the action of the compact subgroup. Finally, Hitchin result has been extended to the non-compact non-unitary case by several other authors [6, 7, 11, 8]. The moduli space of parabolic $K(D)$ pairs is the moduli of holomorphic bundles with a parabolic structure and a holomorphic endomorphism with parabolic residue. It is a Kahler, but not hyper-Kahler manifold, of dimension (in the traceless case) [6]:

$$dim\mathcal{P}^0_\alpha = (2g-2+\mid D\mid)\left(N^2-1\right) . \tag{2.15}$$

It contains as a closed subspace the moduli of Higgs bundles, for which the endomorphism has nilpotent residue. The last one is a hyper-Kahler manifold, of dimension (in the traceless case) [6, 7]:

$$dim\mathcal{N}_\alpha{}^0 = (2g-2)(N^2-1) + 2\sum_{p\in D} f_p\,, \tag{2.16}$$

where $f_p = \frac{1}{2}(N^2 - \sum_i m_i(p)^2)$ and m_i is the multiplicity of the weight α_i associated to the flag structure.

A consequence of this discussion is the following philosophy for approaching the large N limit. Sometimes to solve a problem it may be convenient to embed it into a family. In this case, we want to find the master field by embedding the fluctuations of the theory into a dense family. In fact we consider fluctuations modulo gauge equivalence. It is precisely at this point, after gauging away the gauge group, that the separability of the moduli space requires introducing representations of the fundamental group fiberwise, something very natural in a gauge theory. Since the theory is local, the natural objects are the representations of the fundamental group associated to an arbitrary divisor. This forces the moment maps to be distributions concentrated on the divisor. The requirement of having a dense separable family of gauge-invariant fluctuations has lead us to a very weak topology. This is unavoidable, but the advantage is the integrability of the moduli space.

As a check, let us come back to the functional integral and count, somehow naively, the dimension of the local moduli space. For each point of the divisor we have a traceless hermitian plus a traceless matrix and its complex conjugate, the moment maps, minus a traceless hermitian gauge condition. Following this counting the number of local moduli is $\mid D \mid (N^2-1)$ complex parameters. We should then take into account also the global properties of the torus, but it turns out that on a torus there is no extra contribution, according to Eq.(2.15). The hermitian moment map μ^0 determines a collection of weights via its eigenvalues modulo 2π and a flag structure via its eigenspaces. A holomorphic sheaf is obtained gauging away the field A by means of a complexified gauge transformation (we should stress that complexified gauge transformations are simply a convenient change of variables to describe the moduli in the holomorphic language, not gauge transformations in the functional integral). In this gauge Ψ_z is holomorphic with a certain residue. Generically the residue is conjugated, in the complexified gauge group, to a parabolic element. We have thus a parabolic sheaf together with a parabolic endomorphism Ψ_z .The corresponding moduli space is the one of parabolic $K(D)$ pairs. If the residue of Ψ_z is chosen to be nilpotent the corresponding moduli space is the one of parabolic Higgs bundles [6, 7], that is hyper-Kahler. The inconvenient for controlling the large N limit is, however, that the collective field of $K(D)$ pairs or of Higgs bundles has still a number of components of order N^2. As mentioned in the introduction, a crucial reduction in the entropy is obtained passing to a collective field associated to the Hitchin fibration.

According to Hitchin, by a complexified gauge transformation, Ψ_z can be made holomorphic (with possibly a parabolic or nilpotent residue on the parabolic divisor). The equation

$$Det(\lambda 1 - \Psi_z) = 0 \tag{2.17}$$

determines a spectral cover as a polynomial equation in λ with, as coefficients, N symmetric invariant polynomials in Ψ_z, that are N (meromorphic) [9, 10] differentials. The spectral cover is branched at points where the characteristic polynomial has multiple roots. Since Ψ_z can always be put in triangular form by an extra holomorphic gauge transformation, the eigenspace associated to the first eigenvalue λ_z^1 is one dimensional and depends on z. This defines a holomorphic line bundle. We are thus given a holomorphic line bundle L and a spectral covering π^{-1}. Hitchin shows that Ψ_z can be recovered by L and π^{-1}. Given a section σ of L, the sections of the Higgs bundle in the holomorphic gauge can be recovered as the direct image bundle $\sigma(\pi^{-1})$ [9, 10, 11, 12]. Then the multiplicative action of λ_z^1, thought as a differential on the covering, on the section σ of the line bundle L, determines Ψ_z as an endomorphism of the direct image bundle [9]. This gives Ψ in the holomorphic gauge. To go to a unitary gauge, by making a complexified gauge transformation, we determine A as a solution of the hermitian moment map equation. This leaves still the freedom of making unitary gauge transformations and leaves the choice of an arbitrary hermitian gauge fixing. We have thus recovered the $K(D)$ or Higgs bundle from L and the spectral cover. Let us observe that the sheaf of sections of a holomorphic line bundle, that is equivalent to the line bundle itself, is determined as the kernel of the (0,1) part of an abelian $U(1)$ connection, $a = a_z dz + a_{\bar{z}} d\bar{z}$, [20]. The spectral covers instead are determined by N meromorphic q-differentials, $a_q = a_{z^q} dz^q$. The Hitchin data are therefore those of a vector-like model of complex dimension $N+1$ (N in the traceless case). Yet, the dimension of the moduli space of the Higgs bundles and the one of the line bundles on the spectral coverings, including the moduli of the coverings, are exactly the same. We are now ready to implement this change of variables in the functional integral. We first describe the change of variables at formal level and afterward we exploit the detailed structure of the Jacobian determinant. The QCD partition function can be rewritten as:

$$Z = \int \exp\left[-\Gamma(A,\Psi) + \log \frac{\partial\left(A,\bar{A},\Psi,\bar{\Psi}\right)}{\partial\left(a,\bar{a},a_q,\bar{a}_q\right)}\right] DaD\bar{a}Da_qD\bar{a}_q\,, \tag{2.18}$$

that is, formally, the result looked forward. We want now to give the Jacobian of the change of variables a precise meaning. We describe the computations in a rather sketchy way. We think the functional integral to be stratified by the levels of the moment maps. This is done inserting the obvious resolution of the identity into the

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functional integral, corresponding to the levels, to the over-complete resolution of the levels in terms of coadjoint orbits and to the gauge-fixing:

$$
\begin{aligned}
Z = \lim_{|D|\to\infty} \int & \exp[-\Gamma(A,\Psi)] \times \\
& \times \delta\left(\mu^0 - F_A + i[\Psi,\Psi]\right) \delta\left(\mu - \bar\partial_A \psi\right) \delta\left(\bar\mu - \partial_A \bar\psi\right) \times \\
& \times \delta\left(\mu^0 - \frac{1}{|D|}\sum_p \mu_p^0 \delta_p\right) \delta\left(\mu - \frac{1}{|D|}\sum_p \mu_p \delta_p\right) \delta\left(\bar\mu - \frac{1}{|D|}\sum_p \bar\mu_p \delta_p\right) \times \\
& \times J\left[\mu_0, \mu, \bar\mu\right] \Delta_{FP}\, \delta\left[d^* A + F_{A+\Psi}\right] D\mu^0 D\mu D\bar\mu \times \\
& \times \prod_p d\mu_p^0 d\mu_p d\bar\mu_p d^2 z_p DA_z D\Psi_z DA_{\bar z} D\Psi_{\bar z}\,. \qquad (2.19)
\end{aligned}
$$

The integration over $DA_z D\Psi_z DA_{\bar z} D\Psi_{\bar z}$ gives a combination of functional determinants, T,(the analogue of the analytic torsion [22]) times the Kahler (hyper-Kahler) volume, $\wedge^{max}\omega_{\vec\mu_p}$, associated to the orbits with given level of the moment map, $\vec\mu_p$ (we have grouped the three moment maps into a vector). We now insert the collective field, ρ, associated to the divisor D and the corresponding Jacobian:

$$
Z = \int \exp[-\Gamma(A,\Psi)]\, \frac{J[\vec\mu(\rho)]}{J(\rho)}\, T \wedge^{max} \omega_{\vec\mu(\rho)}\, D\rho\,, \qquad (2.20)
$$

where $\lim_{|D|\to\infty} \int \delta\left(\rho - \frac{1}{|D|}\sum_p \delta_p\right) \prod_p d^2 z_p\, J(\rho) = 1$ and the dependence from the divisor is now expressed through the collective field ρ (this is certainly possible if the moment maps are a symmetric function of the points of the divisor; in particular if $\vec\mu_p = \vec\mu$ for all p, as it is expected for a translationally invariant ground state). Then, we change variables to the Hitchin fibration in terms of the the symplectic volume form $\wedge^{max}\omega_\rho$ associated to the symplectic structure on the cotangent space of rank 1 bundles on the spectral covering:

$$
\begin{aligned}
Z = \int & \exp[-\Gamma(A,\Psi)] \times \\
& \times \frac{J[\vec\mu(\rho)]}{J(\rho)}\, T\, \frac{\wedge^{max}\omega_{\vec\mu(\rho)}}{\wedge^{max}\omega_\rho} \times \wedge^{max}\omega_\rho\, D\rho\,, \qquad (2.21)
\end{aligned}
$$

where ω_ρ can be chosen to be a symplectic form on rank 1 Higgs bundles on the spectral covers $\pi^{-1}(T_f^2)$, with coordinates $(a(\rho), \lambda^1(\rho))$:

$$
\omega_\rho = \int_{\pi^{-1}(T_f^2)} \delta a(\rho) \wedge \delta \bar a(\rho) + \delta\lambda^1(\rho) \wedge \delta\bar\lambda^1(\rho)\,. \qquad (2.22)
$$

Finally, we represent $\wedge^{max}\omega_\rho$ as the $U(1)$ analytic torsion [22] on the coverings times the volume of the coverings:

$$
\begin{aligned}
\wedge^{max}\omega_\rho = \delta\left(F_{a(\rho)}\right)\Big|_{\pi^{-1}(T_f^2)} & \delta[d^* a(\rho)] \Delta_{FP}(a(\rho)) Da(\rho) D\bar a(\rho) \times \\
& \times \frac{\partial(\lambda(\rho), \bar\lambda(\rho))}{\partial\left(a_q(\rho), \bar a_q(\rho)\right)} Da_q(\rho) D\bar a_q(\rho)\,, \qquad (2.23)
\end{aligned}
$$

where, in the volume form for the coverings, we have changed variables, from the eigenvalues λ of Ψ_z, to the meromorphic differentials a_{z^q} $(= Tr[\Psi_z{}^q])$, holomorphic away from the divisor D. Notice that the entropy, associated to the measure in Eq. (2.23), is of order N, since only N fields, a_{z^q}, living on T_f^2 and one, a, living on $\pi^{-1}(T_f^2)$ (that is a N-sheeted covering) are integrated. Actually we should take into account also the integration over the collective field of the divisor ρ in Eq. (2.21), but this is still a contribution of order one. This completes our sketchy evaluation of the Jacobian determinant to the collective field of the Hitchin fibration.

3. Conclusions

To summarize, we represent the QCD partition function, by means of a fiberwise duality transformation on the base torus, as a functional integral on $T^*\mathcal{A}$ fiberwise:

$$\begin{aligned}
Z = \int \exp\left[-\frac{N}{2\lambda}\int Tr\left(F_A^2 + (d^*A)^2 + (\partial_\perp A)^2\right) + (\text{non-local terms})\, d^4x\right] \times \\
\times\, Det\left[-\Delta_A - i\, ad_{F_{A+\Psi}}\right]^{-1} \Delta_D(A,\Psi) \times \\
\times\, \Delta_{FP}\delta[d^*A + F_{A+\Psi}] DA_z D\Psi_z DA_{\bar z} D\Psi_{\bar z} = \\
= \int \exp[-\Gamma(A,\Psi)] DA_z D\Psi_z DA_{\bar z} D\Psi_{\bar z}\,. \qquad (3.1)
\end{aligned}$$

Then, we choose a dense subset of $T^*\mathcal{A}$ (dense in the large N limit), that admits a separable Kahler (hyper-Kahler) quotient under the action of the gauge group. This dense subset turns out to be the inductive limit of the sheaves of $K(D)$ pairs (Higgs bundles) with arbitrary parabolic divisor D. Thereafter, on this dense subset, we change variables from $(A_z, \Psi_z, A_{\bar z}, \Psi_{\bar z})$ to $(a_z, a_{z^q}, a_{\bar z}, a_{\bar z^q})$, where $(a_z, a_{\bar z})$ is a $U(1)$ connection on a N- sheeted covering and $(a_{z^q}; q = 1, ..., N)$ are N gauge-invariant meromorphic q-differentials, holomorphic away from D, fiberwise over the base torus.

Hence, since our theory is now labelled by a $U(1)$ connection plus a set of N-chiral and anti-chiral fields of increasing spin, up to N, it turns out that the effective action of QCD, in the large N limit, reads:

$$\Gamma_{eff}(A,\Psi) = \Gamma(A,\Psi) + \log\frac{\partial(a,\bar a, a_q, \bar a_q)}{\partial(A,\bar A,\Psi,\bar\Psi)}\,. \qquad (3.2)$$

The effective action Γ_{eff}, as a functional of a parabolic Higgs bundle, need not to be finite, because of the singularities of the connection and Higgs field on the parabolic divisor. However, this is not unexpected. The changes of variable, that we performed in the functional integral to get the effective action, are by no means a substitute of regularization and renormalization. In this respect we are in the same situation as for the Migdal-Makeenko equation [23], that needs to be regularized and renormalized. What is needed here is a regularization of the theory that respects as most as possible the moduli structure of the parabolic Higgs bundles and of the Hitchin fibration, that is, the gauge structure of the theory. Surprisingly, such regularization exists and it consists in analytically continuing the fields (A, Ψ) from Euclidean to Minkowskian space (sic!) [24] (p.186). This gives every pole, in the (x_+, x_-) Minkowskian coordinates, a $i\epsilon$ part, moving away the singularity from the real integration path. The divergences are then obtained in the $\epsilon \to 0$ limit and ϵ is our regularization parameter.

An important point is that the master field should be gauge equivalent to a constant [25]. In the present approach, instead, it is described as a collective field of distributions. Yet, it is conceivable that this collective field converges to a constant, in the large N limit, in the sense of the distributions. In fact, the weak topology is often the strongest topology that may be introduced in field theory, as the asymptotic weak convergence of interacting fields to free fields, in scattering theory, shows.

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